Demostrar que $\{A,\{B,C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} = 0$, donde $\{x,y\}$ es el Corchete de Poisson.

Solución.- Tenemos por separado:

(*) $$\{A,\{B,C\}\} = \left\{A, \left(\frac{\partial B}{\partial z}\right)^T \cdot J \cdot \frac{\partial C}{\partial z}\right\} = \left(\frac{\partial A}{\partial z}\right)^T \cdot J \cdot \frac{\partial}{\partial z}\left[\left(\frac{\partial B}{\partial z}\right)^T \cdot J \cdot \frac{\partial C}{\partial z}\right] =$$

$$= \left(\frac{\partial A}{\partial z}\right)^T J \cdot \left\{ \left[\frac{\partial}{\partial z}\left(\frac{\partial B}{\partial z}\right)^T\right] \cdot J \cdot \frac{\partial C}{\partial z} + \left(\frac{\partial B}{\partial z}\right)^T \frac{\partial}{\partial z}\left(J \cdot \frac{\partial C}{\partial z}\right)\right\} =$$

$$= \left(\frac{\partial A}{\partial z}\right)^T \cdot J \cdot \left\{ \left[\frac{\partial}{\partial z}\left(\frac{\partial B}{\partial z}\right)^T\right] \cdot J \cdot \frac{\partial C}{\partial z} + \left(\frac{\partial B}{\partial z}\right)^T \cdot J \cdot \frac{\partial}{\partial z}\left(\frac{\partial C}{\partial z}\right)\right\} =$$

$$= \left(\frac{\partial A}{\partial z}\right)^T \cdot J \cdot \left[\frac{\partial}{\partial z}\left(\frac{\partial B}{\partial z}\right)^T\right] \cdot J \cdot \frac{\partial C}{\partial z} + \left(\frac{\partial A}{\partial z}\right)^T \cdot J \cdot \left(\frac{\partial B}{\partial z}\right)^T \cdot J \cdot \frac{\partial}{\partial z}\left(\frac{\partial C}{\partial z}\right) =$$

$$= \left\{A, \left(\frac{\partial B}{\partial z}\right)^T\right\} \cdot J \cdot \frac{\partial C}{\partial z} + \left(\frac{\partial A}{\partial z}\right)^T \cdot J \cdot \left\{B, \frac{\partial C}{\partial z}\right\}. \qquad [1]$$

(*) $$\{B,\{C,A\}\} = \left\{B, \left(\frac{\partial C}{\partial z}\right)^T\right\} \cdot J \cdot \frac{\partial A}{\partial z} + \left(\frac{\partial B}{\partial z}\right)^T \cdot J \cdot \left\{C, \frac{\partial A}{\partial z}\right\} \qquad [2]$$

(*) $$\{C,\{A,B\}\} = \left\{C, \left(\frac{\partial A}{\partial z}\right)^T\right\} \cdot J \cdot \frac{\partial B}{\partial z} + \left(\frac{\partial C}{\partial z}\right)^T \cdot J \cdot \left\{A, \frac{\partial B}{\partial z}\right\} \qquad [3]$$

Sumamos:

$$\{A,\{B,C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} = \boxed{\left\{A, \left(\frac{\partial B}{\partial z}\right)^T\right\} J \frac{\partial C}{\partial z}} + \boxed{\left(\frac{\partial A}{\partial z}\right)^T J \left\{B, \frac{\partial C}{\partial z}\right\}} + \boxed{\left\{C, \left(\frac{\partial A}{\partial z}\right)^T\right\} J \frac{\partial B}{\partial z}} +$$

$$+ \boxed{\left(\frac{\partial C}{\partial z}\right)^T J \cdot \left\{A, \frac{\partial B}{\partial z}\right\}} + \boxed{\left\{B, \left(\frac{\partial C}{\partial z}\right)^T\right\} J \frac{\partial A}{\partial z}} + \boxed{\left(\frac{\partial B}{\partial z}\right)^T J \left\{C, \frac{\partial A}{\partial z}\right\}} =$$

$$= \left\{A, \left(\frac{\partial B}{\partial z}\right)^T\right\} J \frac{\partial C}{\partial z} + \left(\frac{\partial C}{\partial z}\right)^T \cdot J \cdot \left\{A, \frac{\partial B}{\partial z}\right\} + \left(\frac{\partial A}{\partial z}\right)^T J \left\{B, \frac{\partial C}{\partial z}\right\} + \left\{B, \left(\frac{\partial C}{\partial z}\right)^T\right\} J \frac{\partial A}{\partial z} +$$

$$+ \left\{C, \left(\frac{\partial A}{\partial z}\right)^T\right\} J \frac{\partial B}{\partial z} + \left(\frac{\partial B}{\partial z}\right)^T \cdot J \cdot \left\{C, \frac{\partial A}{\partial z}\right\} =$$

$$= \left\{A, \left(\frac{\partial B}{\partial z}\right)^T\right\} J \frac{\partial C}{\partial z} - \left\{A, \frac{\partial B}{\partial z}\right\} J \frac{\partial C}{\partial z} + \left\{B, \left(\frac{\partial C}{\partial z}\right)^T\right\} J \frac{\partial A}{\partial z} - \left\{B, \frac{\partial C}{\partial z}\right\} J \frac{\partial A}{\partial z} +$$

$$+ \left\{C, \left(\frac{\partial A}{\partial z}\right)^T\right\} J \frac{\partial B}{\partial z} - \left\{C, \frac{\partial A}{\partial z}\right\} J \cdot \frac{\partial B}{\partial z} =$$

$$= \left[\left\{A, \left(\frac{\partial B}{\partial z}\right)^T\right\} - \left\{A, \frac{\partial B}{\partial z}\right\}\right] J \frac{\partial C}{\partial z} + \left[\left\{B, \frac{\partial C}{\partial z}\right\} - \left\{B, \left(\frac{\partial C}{\partial z}\right)^T\right\}\right] J \cdot \frac{\partial A}{\partial z} + \left[\left\{C, \frac{\partial A}{\partial z}\right\} - \left\{C, \left(\frac{\partial A}{\partial z}\right)^T\right\}\right] J \frac{\partial B}{\partial z} \qquad [4]$$

Veamos que, por definición,

$$\left\{A, \frac{\partial B}{\partial z}\right\} = \sum_{i=1}^{n} \frac{\partial A}{\partial q_i}\frac{\partial\left(\frac{\partial B}{\partial z}\right)}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial\left(\frac{\partial B}{\partial z}\right)}{\partial q_i}$$

donde, si $B = B(q_1, \dots, q_n, p_1, \dots, p_n)$ entonces

$$\frac{\partial B}{\partial z} = \left(\frac{\partial B}{\partial q_1} \dots \frac{\partial B}{\partial q_n} \; \frac{\partial B}{\partial p_1} \dots \frac{\partial B}{\partial p_n}\right).$$

y tendremos que,

$$\frac{\partial}{\partial q_i}\left(\frac{\partial B}{\partial z}\right) = \left(0 \dots 0 \; \frac{\partial^2 B}{\partial q_i^2} \; 0 \dots 0\right)$$

De manera análoga,

$$\left(\frac{\partial B}{\partial z}\right)^T = \begin{pmatrix} \partial B/\partial q_1 \\ \vdots \\ \partial B/\partial q_n \\ \partial B/\partial p_1 \\ \vdots \\ \partial B/\partial p_n \end{pmatrix} \Rightarrow \frac{\partial}{\partial q_i}\left(\frac{\partial B}{\partial z}\right)^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \partial^2 B/\partial q_i^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Por tanto podemos asegurar que,

$$\left\{A, \frac{\partial B}{\partial z}\right\} = \left\{A, \left(\frac{\partial B}{\partial z}\right)^T\right\} = \sum_{i=1}^{n}\left(\frac{\partial A}{\partial q_i}\frac{\partial^2 B}{\partial p_i^2} - \frac{\partial A}{\partial p_i}\frac{\partial^2 B}{\partial q_i^2}\right)$$

que al llevar esta conclusión a [4] resulta,

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} =$$

$$= \left[\underbrace{\left\{A, \left(\frac{\partial B}{\partial z}\right)^T\right\} - \left\{A, \frac{\partial B}{\partial z}\right\}}_{0}\right] J \frac{\partial C}{\partial z} + \left[\underbrace{\left\{B, \frac{\partial C}{\partial z}\right\} - \left\{B, \left(\frac{\partial C}{\partial z}\right)^T\right\}}_{0}\right] J \frac{\partial A}{\partial z} + \left[\underbrace{\left\{C, \frac{\partial A}{\partial z}\right\} - \left\{C, \left(\frac{\partial A}{\partial z}\right)^T\right\}}_{0}\right] J \frac{\partial B}{\partial z} =$$

$$= 0\, J \frac{\partial C}{\partial z} + 0\, J \frac{\partial A}{\partial z} + 0\, J \frac{\partial B}{\partial z} = 0 + 0 + 0 = 0,$$

como queríamos demostrar.